Aerodynamics of Wings in Supersonic Shear Flow

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A lifting surface theory is developed for wings in a nonuniform supersonic parallel stream whose velocity and density vary in the vertical direction. An integral equation for the pressure load on the wing is obtained by applying Laplace and Fourier transformations to the linearized equations of inviscid flow. An explicit solution is derived for wings with supersonic edges. Numerical results show the effects of stream nonuniformity on the lift of delta wings with supersonic leading edges in a jet stream, a wake stream, and monotonic sheared streams.

Introduction

THE objective of this study is to develop a method for calculating the flow and the lift forces on a wing in a nonuniform parallel supersonic stream whose velocity and density vary in the vertical direction. The aerodynamics of wings in a nonuniform stream can be of practical interest since it relates to flight in a sheared wind, wake, or in a jet. It can become particularly important at supersonic speeds, since a wing is then often situated behind a curved shock wave generated by a forebody or a canard surface, so that the flow ahead of the wing is not uniform.

The literature on wings in supersonic nonuniform stream is scarce. Williams et al.[1] obtained numerical solutions for wings in a thin shear layer simulating a turbulent boundary layer with supersonic outer velocity. Hanin and Barsony-Nagy[2] recently studied two-dimensional flow over an airfoil in a supersonic nonuniform stream with an arbitrary Mach number profile. In a previous paper[3] they gave a theory of slender wings in a nonuniform stream, valid for small aspect ratios at subsonic and low supersonic velocities.

In the present work, a lifting surface integral equation is obtained for three-dimensional wings in a nonuniform parallel stream having an arbitrary supersonic Mach number profile $M(z)$. It is assumed only that the profile has a finite second derivative $M''(0)$ at the wing plane ($z=0$) and that the Mach number has finite limits $M(\pm\infty)$ at large distances above and below the wing. The flow perturbations due to the wing are considered to be small and inviscid, so that they are governed by the Euler equations of compressible flow, linearized for small deviations from the nonuniform supersonic stream. To solve the equations, a streamwise Laplace transformation and a spanwise Fourier transformation are made and a fundamental transform solution is determined. An integral equation for the pressure load on the wing is then deduced by inverting the transform solution and applying the flow tangency condition on the wing. In the case of a wing with supersonic edges, an explicit solution of the integral equation is obtained.

Numerical results are presented for plane delta wings with supersonic leading edges and several Mach number profiles representing a jet stream, a wake stream, and monotonic sheared streams. A brief parametric study is made of the effects of stream nonuniformity on the lift.

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Supersonic Lifting Surface Theory

The wing under consideration is set at a small angle of attack in a parallel nonuniform supersonic stream whose velocity, density, and temperature vary in the vertical direction. We take the axes x, y, and z in the streamwise, spanwise, and vertical directions, respectively, with the origin at the wing apex. The profiles of velocity, density, speed of sound, and Mach number in the undisturbed stream are denoted by $U(z)$, $\rho(z)$, $a(z)$, and $M(z)=U(z)/a(z)$. Since the stream is parallel, the undisturbed pressure does not vary with z. The wing is situated near the $z=0$ plane and is assumed to have a negligible thickness.

The inviscid flow perturbations generated by the wing are governed by the Euler equations of momentum, continuity, and energy of a perfect gas, linearized for small deviations from the undisturbed nonuniform stream. The linearized equations of flow give a single equation for the pressure perturbation p,

$$[M^2(z)-1]p_{xx}-p_{yy}-p_{zz}+[2M'(z)/M(z)]p_z=0 \quad (1)$$

where the subscripts denote partial derivatives and $M'(z)=\mathrm{d}M/\mathrm{d}z$. This equation was given first by Lighthill[4] in a different context. It was used in our studies of subsonic wings and of supersonic two-dimensional airfoils in nonuniform flow.[2,5]

In the present study the stream is supersonic,

$$M(z)>1 \quad (2)$$

For a wing without thickness, the linearized boundary condition on the wing surface is

$$w(x,y,\pm 0)=-U(0)\alpha(x,y) \quad (3)$$

where α denotes the local angle of attack and w the upward velocity component. In view of the linearized z momentum equation

$$\rho(z)U(z)w_x+p_z=0 \quad (4)$$

the boundary condition on the wing becomes

$$\int_0^x p_z(\xi,y,\pm 0)\,\mathrm{d}\xi=\rho(0)U^2(0)\alpha(x,y) \quad (5)$$

At large distances from the wing, the flow must satisfy the "radiation" condition, which requires that the perturbations behave like an outgoing wave. There are no perturbations in the upstream region ahead of the characteristic surfaces emerging from the wing.

Our aim is to deduce an equation for the pressure load $\ell(x,y)$ on the wing, i.e., the pressure difference between the lower and upper sides,

$$\ell(x,y) = p(x,y,-0) - p(x,y,+0) \tag{6}$$

To solve Eq. (1), we apply a Laplace transformation with respect to x and then a Fourier transformation with respect to y. The double transform $\tilde{f}$ of a function f is defined by

$$\tilde{f}(s,k,z) = \int_{-\infty}^{\infty} e^{-iky} \int_0^{\infty} e^{-sx} f(x,y,z)\,dx dy \tag{7}$$

where k is real and s complex with a positive real part. Equations (1), (5), and (6) become

$$\tilde{p}_{zz} - [2M'(z)/M(z)]\tilde{p}_z - [(M^2(z)-1)s^2+k^2]\tilde{p} = 0 \tag{8}$$

$$\tilde{p}_z(s,k,+0) - \tilde{p}_z(s,k,-0) = 0 \tag{9}$$

$$\tilde{p}(s,k,+0) - \tilde{p}(s,k,-0) = -\tilde{\ell}(s,k) \tag{10}$$

and from the radiation condition at large vertical distances from the wing we have

$$\tilde{p}(s,k,\pm\infty) = 0 \tag{11}$$

An integral equation for the pressure load $\ell(x,y)$ can be obtained by solving Eq. (8) with the boundary conditions stated in Eqs. (9-11), inverting the transform to get a relation between p and ℓ, and then applying the flow tangency condition [Eq. (5)].

Since the solution of Eq. (8) with the boundary conditions given in Eqs. (9-11) depends linearly on the pressure load transform $\tilde{\ell}$, we have

$$\tilde{p} = -\tilde{\ell}(s,k)\tilde{P}(s,k,z) \tag{12}$$

where $\tilde{P}$ is the solution of Eq. (8) satisfying the boundary conditions

$$\tilde{P}(s,k,+0) - \tilde{P}(s,k,-0) = 1 \tag{13a}$$

$$\tilde{P}_z(s,k,+0) - \tilde{P}_z(s,k,-0) = 0 \tag{13b}$$

$$\tilde{P}(s,k,\pm\infty) = 0 \tag{13c}$$

The function $\tilde{P}$ and its inverse $P(x,y,z)$ may be regarded as the fundamental solution for supersonic nonuniform stream, in analogy to the fundamental solution for incompressible flow introduced by Lighthill.[6] The functions $\tilde{P}$ and P depend only on the Mach number profile $M(z)$ of the stream and do not depend on the wing geometry or the angle of attack. The fundamental transform solution $\tilde{P}$ can be calculated by computing two solutions of Eq. (8) that satisfy respectively the conditions

$$\tilde{P}_1(s,k,+\infty) = 0 \qquad \tilde{P}_2(s,k,-\infty) = 0$$

then taking

$$\begin{aligned}\tilde{P} &= C_1(s,k)\tilde{P}_1(s,k,z) \quad \text{for } z>0 \\ &= C_2(s,k)\tilde{P}_2(s,k,z) \quad \text{for } z<0\end{aligned} \tag{14}$$

and determining C_1 and C_2 from the boundary conditions [Eqs. (13a) and (13b)]. It can be shown that a sufficient condition for the existence of the solution is the convergence of the integral

$$\int_{-\infty}^{\infty} |M'(z)|\,dz$$

The pressure perturbation in the flowfield is expressed in terms of the pressure load on the wing and the fundamental solution by applying the convolution theorem to Eq. (12), which gives

$$p(x,y,z) = -\int_0^x \int_{-\infty}^{\infty} \ell(x',y')P(x-x',y-y',z)\,dy'dx' \tag{15}$$

An integral equation for the pressure load is now obtained by utilizing the boundary condition on the wing as stated in Eq. (5). This gives

$$\int_0^x \int_{-\infty}^{\infty} \ell(x',y')N(x-x',y-y')dy'dx' = -\rho(0)U^2(0)\alpha(x,y) \tag{16}$$

where

$$N(x,y) = \int_0^x P_z(\xi,y,0)\,d\xi \tag{17}$$

The kernel N has singularities that must be taken into account in solving the integral equation. To find the singularities, we expand the transformed kernel $\tilde{N}$ asymptotically for large $|k|$ and $|s|$, getting

$$s\tilde{N} = \tilde{P}_z(s,k,0) = -\tfrac{1}{2}(\beta^2 s^2+k^2)^{1/2} + \tfrac{1}{2}\sigma(\beta^2 s^2+k^2)^{-1/2} + 0[(\beta^2 s^2+k^2)^{-3/2}] \tag{18}$$

where we denote

$$\beta = \sqrt{M^2(0)-1} \tag{19}$$

$$\sigma = M''(0)/2M(0) \tag{20}$$

Inverting the terms[7] shows that

$$2\pi N_x(x,y) = [-\beta^2(x^2-\beta^2y^2)^{-3/2} + \sigma(x^2-\beta^2y^2)^{-1/2}]H(x^2-\beta^2y^2) + Q(x,y) \tag{21}$$

where Q is a bounded continuous function that vanishes for $x^2 < \beta^2 y^2$, and H is the Heaviside unit function

$$\begin{aligned}H(\lambda) &= 1, \quad \lambda \geq 0 \\ &= 0, \quad \lambda > 0\end{aligned} \tag{22}$$

The kernel N is now found by integrating Eq. (21), which gives

$$N(x,y) = \frac{1}{2\pi}[xy^{-2}(x^2-\beta^2y^2)^{-1/2} + \sigma\ln[x + (x^2-\beta^2y^2)^{1/2}] + R(x,y)]H(x^2-\beta^2y^2) \tag{23}$$

where R is the continuous bounded function

$$R(x,y) = \int_{\beta|y|}^{x} Q(\xi,y)\,d\xi \tag{24}$$

It is seen that the kernel $N(x-x', y-y')$ of Eq. (16) has singularities at the line $y-y'=0$ and the two Mach lines

$$x-x' \pm \beta(y-y') = 0$$

The resulting integral equation for the pressure load on a wing in a supersonic nonuniform stream is

$$\int_0^x \int_{y-(x-x')/\beta}^{y+(x-x')/\beta} \ell(x',y')\left[\frac{x-x'}{(y-y')^2 r'} + \sigma\ln(x-x'+r') + R(x-x',y-y')\right]dy'dx' = -2\pi\rho(0)U^2(0)\alpha(x,y) \tag{25}$$

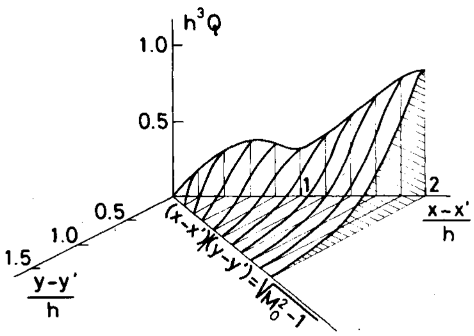

Fig. 1 Kernel function Q for jet stream, $M_0 = 2$, $M_0/M_I = 1.5$.

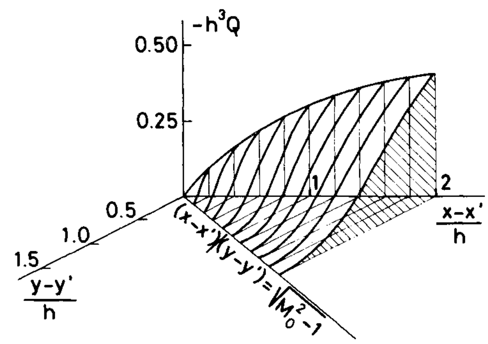

Fig. 2 Kernel function Q for linear sheared stream, $M_0 = 2$, $M_I/M_2 = 2$.

where

$$r' = \sqrt{(x-x')^2 - \beta^2 (y-y')^2} \tag{26}$$

In Eq. (25), the integration is carried out over the wing planform only, within the forward Mach angle of the point (x,y), since outside the wing $\ell = 0$. The improper integral involving the $(y-y')^{-2}$ singularity in the first term of the kernel must be taken as the principal value.

Equation (25) holds for arbitrary wing planforms and can be regarded as a lifting surface integral equation for a nonuniform supersonic stream. The first term of its kernel coincides with the classical supersonic lifting surface equation for uniform stream.[8] The two other terms are due to the stream nonuniformity.

The bounded term R of the kernel can be computed by solving numerically Eq. (8) for the fundamental transform solution $\tilde{P}$ as indicated in Eq. (14), then calculating $\tilde{N} = \tilde{P}_z(s,k,0)/s$, inverting numerically to get N, and finally subtracting the two singular terms in accordance with Eq. (23). A better procedure, which eliminates infinities from the computation, is to subtract first the transforms of the two singular terms from the solution $\tilde{P}_z$ and then invert to obtain the bounded functions Q and R. The numerical inversion of the Fourier transform can be done by using a standard integration method. The Laplace transform can be inverted by taking the complex integration path in the s plane along the line $Re(s) = \text{const} > 0$ and using the same standard method to compute the ensuing Fourier integral.

The function R, as well and N and Q, depends only on the Mach number profile $M(z)$ of the stream and does not depend on the wing planform or the angle of attack.

Solution for Wings with Supersonic Edges

In the case of wings having supersonic leading and trailing edges, an explicit solution of the lifting surface equation can be found. For this purpose we return to Eq. (16) and note that the equation holds in the entire half-plane $x > 0$, provided that $\alpha(x,y)$ outside the wing is taken as the local downwash angle. Applying the Laplace and Fourier transformations to Eq. (16) and using the convolution theorem gives‡

$$\tilde{\ell} = -\rho(0)\, U^2(0)\, \tilde{\alpha}(s,k) / \tilde{N}(s,k) \tag{27}$$

From the asymptotic expansion in Eq. (18) we have

$$\frac{1}{s\tilde{N}} = -2(\beta^2 s^2 + k^2)^{-1/2} - 2\sigma(\beta^2 s^2 + k^2)^{-3/2} - \frac{2}{\pi}\tilde{T}(s,k) \tag{28}$$

where

$$\tilde{T} = 0[(\beta^2 s^2 + k^2)^{-5/2}]$$

The pressure load on the wing, or rather its chordwise integral

$$L = \int_0^x \ell(\xi, y)\, d\xi \tag{29}$$

is now obtained by inverting the terms of $\tilde{L} = \tilde{\ell}/s$ with the aid of the convolution theorem. The result is

$$L(x,y) = \frac{2}{\pi}\rho(0)\, U^2(0) \int_0^x \int_{y-(x-x')/\beta}^{y+(x-x')/\beta} \alpha(x',y') \left[\frac{1}{r'} + \frac{\sigma}{\beta^2} r' + T(x-x', y-y')\right] dy'\, dx' \tag{30}$$

where T is a bounded function that can be calculated from the fundamental solution $\tilde{P}$ using Eqs. (28) and (17). When the wing has supersonic edges, we have in Eq. (30) $\alpha = 0$ ahead of the leading edge, and for a point (x,y) on the wing the trailing edge does not enter the integration region. Thus, Eq. (30) relates the pressure load directly to the angle-of-attack distribution of the wing.

The first term in Eq. (30) coincides with the well-known result for a uniform stream. The second and the third terms have the relative orders of magnitude of c^2/h^2 and c^4/h^4, respectively, where c is the wing chord and h the vertical extent of stream nonuniformity.

Numerical Results

To obtain an insight into the properties of wings in nonuniform supersonic streams, the fundamental solution and the related kernel functions were computed for several Mach number profiles representing supersonic flight in a jet, in a wake, in a linearly sheared stream, and in a nonlinear monotonic sheared stream. The lift of plane delta wings with supersonic leading edges was then found for these profiles by applying Eq. (30).

The jet stream and wake stream profiles were described by

$$M(z) = M_I + (M_0 - M_I)e^{-z^2/h^2} \tag{31}$$

where the wing location $z = 0$ is taken at the symmetry plane of the stream, $M_0 = M(0)$ is the local Mach number there, M_I is the outer Mach number, and h is related to the vertical extent of the jet or wake.

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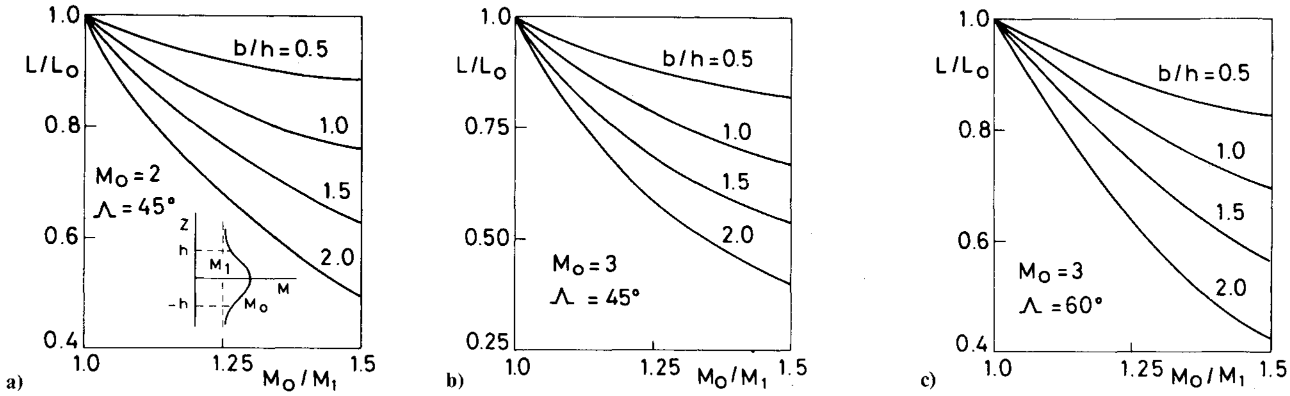

Fig. 3 Lift of delta wings in jet stream.

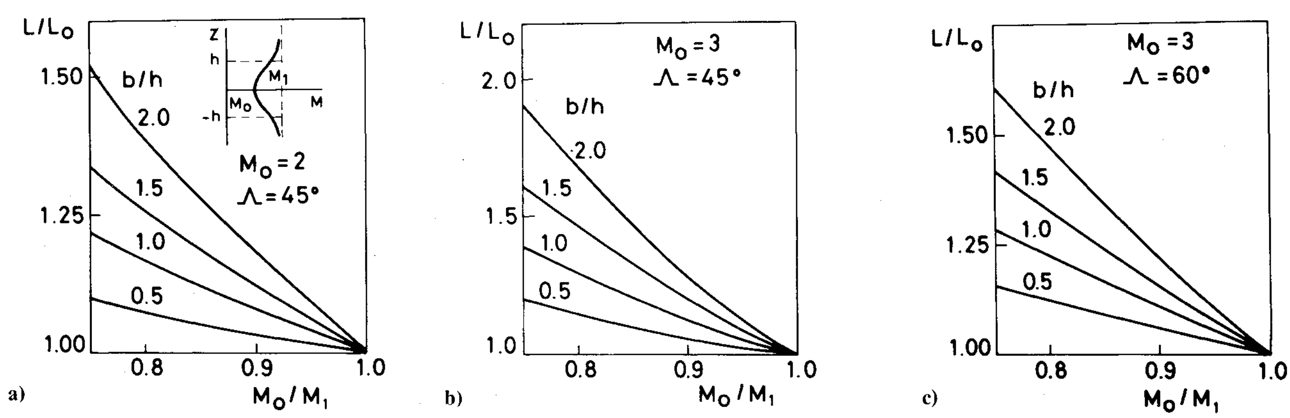

Fig. 4 Lift of delta wings in wake stream.

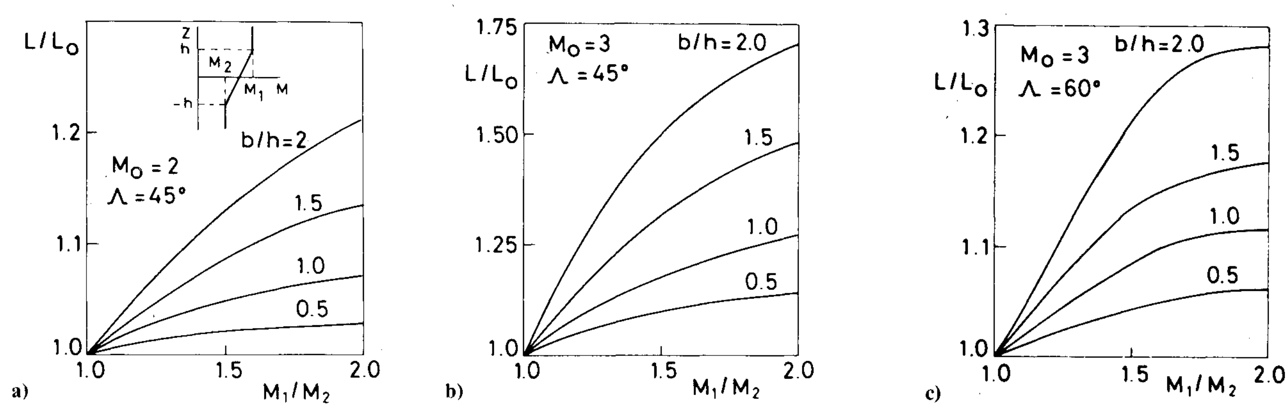

Fig. 5 Lift of delta wings in linear sheared stream.

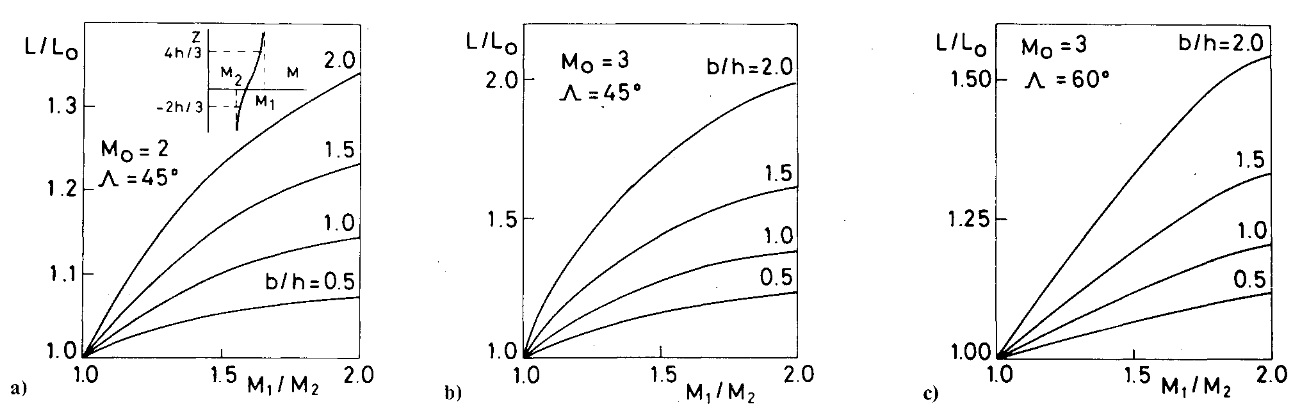

Fig. 6 Lift of delta wings in nonlinear sheared stream.

The linearly sheared stream profile, with the wing placed at the middle plane, is given by

$$\begin{aligned} M &= M_1 && \text{for } z>h \\ &= \tfrac{1}{2}(M_1+M_2)+\tfrac{1}{2}(M_1-M_2)(z/h) && \text{for } -h<z<h \\ &= M_2 && \text{for } z<-h \end{aligned} \quad (32)$$

The nonlinear monotonic sheared profile was chosen as

$$M=\tfrac{1}{2}(M_1+M_2)+\tfrac{1}{2}(M_1-M_2)\tanh[(z-z_0)/h] \quad (33)$$

and we took $z_0=h/3$, which placed the wing below the middle plane of the shear layer.

The fundamental transform solution $\tilde{P}$ for these profiles was computed using Eq. (14). The kernel function Q and the resolvent T were then calculated by subtracting the singular terms in accordance with Eqs. (18) and (28) and inverting the transforms by numerical integrations. The densities of the points (s,k) for which the solution $\tilde{P}$ was computed were increased until the resulting kernel Q did not change appreciably. A substantial reduction of the required density of solution points at large $|s|$ and $|k|$ was achieved by interpolating the solution with the aid of an expansion in negative powers of $(\beta^2 s^2+k^2)^{1/2}$. The number of the required solution points (s,k) was then about 15×15 to 25×25, depending on the Mach number profile.

Figures 1 and 2 show the kernel function Q for two profiles. We observe that the computed values of Q vanish on the Mach angle $x-x'=\beta|y-y'|$ and outside, as is expected. For each profile, the nondimensional kernel h^2Q is a function of the dimensionless distances $(x-x')/h$ and $(y-y')/h$, and depends also on the Mach number at the wing plane M_0 and on the Mach number ratio across the stream M_0/M_1 or M_1/M_2.

The calculated lift of plane delta wings with supersonic leading edges in nonuniform streams is presented in Figs. 3-6. The quantity shown is the ratio L/L_0, where L is the lift in nonuniform stream and L_0 denotes the lift of the same wing at the same angle of attack in a uniform stream having the velocity $U(0)$, density $\rho(0)$ and Mach number $M(0)$. For plane delta wings with supersonic edges

$$L_0=\tfrac{1}{2}\rho(0)\,U^2(0)\,b^2\tan\Lambda[\alpha/\sqrt{M^2(0)-1}] \quad (34)$$

where b is the wing span, Λ the angle of sweep, and α the angle of attack. The lift ratio L/L_0 depends on the Mach number $M_0=M(0)$ at the wing plane, the angle of sweep, the Mach number ratio across the stream M_0/M_1 or M_1/M_2, and the ratio of wing span to stream scale b/h.

Significant effects of stream nonuniformity on the lift are found in a wide range of the parameters. The variations of L/L_0 away from unity become stronger when the scale ratio b/h increases, the angle of sweep Λ decreases, and the Mach number at the wing plane M_0 increases. The lift ratio varies more strongly in the jet stream, the wake stream, and the nonlinear sheared stream than in the linearly sheared stream. This is due to the direct effect of the second derivative of the Mach number profile $M''(0)$ on Eqs. (25) and (30) through the parameter σ.

It may be expected that for large values of the scale ratio b/h the lift in a jet or wake stream is determined mainly by the outer velocity and density, so that we should have $L/L_0\approx(M_1/M_0)^2$ when $b/h\gg 1$. The lines for $b/h=2.0$ are already not far from this behavior. On the other hand, at $b/h=0$ we have $L/L_0=1$, since then the stream becomes uniform. Figures 3-6 show the behavior in the intermediate range in which the lift ratio L/L_0 varies considerably with the scale ratio b/h.

Conclusions

A method has been developed for calculating the flow and the pressures on a wing in a supersonic nonuniform stream whose velocity and density vary in the vertical direction. A lifting surface integral equation has been obtained for the pressure load on a wing with arbitrary planform and zero thickness, set at an angle of attack in a supersonic stream having an arbitrary Mach number profile. In the case of wings with supersonic edges, an explicit solution has been derived. The leading terms in the kernel of the lifting surface equation and in the resolvent for wings with supersonic edges coincide with the classical theory of wings in uniform stream. The remaining terms are due to the stream nonuniformity and can be computed by solving numerically an ordinary differential equation and inverting Fourier and Laplace transforms.

Numerical results have been presented for plane delta wings with supersonic edges and several stream Mach number profiles. Significant effects of the stream nonuniformity on the lift have been found in a wide range of the parameters. The variations of lift with the Mach number ratio of the stream become stronger when the Mach number at the wing plane increases, the angle of sweep decreases, and the ratio of wing span to stream nonuniformity scale increases.

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